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# On the all-order epsilon-expansion of generalized hypergeometric functions with integer values of parameters

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Abstract:
We continue our study of the construction of analytical coefficients of the epsilon-expansion of hypergeometric functions and their connection with Feynman diagrams. In this paper, we apply the approach of obtaining iterated solutions to the differential equations associated with hypergeometric functions to prove the following result:

**Theorem 1:**
The epsilon-expansion of a generalized hypergeometric function with integer values of parameters,

$$ {}_pF_{p-1}(I_1+a_1\varepsilon,\cdots,I_p+a_p\varepsilon;I_{p+1}+b_1\varepsilon,\cdots,I_{2p-1}+b_{p-1};z) \, , $$

is expressible in terms of generalized polylogarithms with coefficients that are ratios of polynomials.

The method used in this proof provides an efficient algorithm for calculating of the higher-order coefficients of Laurent expansion.

Keywords: Differential and Algebraic Geometry, NLO Computations.

## Contents



# 1. Introduction

Hypergeometric functions are useful in the evaluation of Feynman diagrams. See, for example, Ref. [1] for a review of how these functions arise. In this paper, we will be concerned with the manipulation of hypergeometric functions [2, 3], by which we understand specifically

(1) the reduction of the original function to a minimal set of basis functions,

(2) the construction of the all-order $\varepsilon$-expansion of basis functions.

The $\varepsilon$-expansion refers to the Laurent expansion of hypergeometric functions about rational values of their parameters in terms of known functions or perhaps new types of functions. In the latter case, the problem remains to identify the full set of functions which must be invented to construct this expansion for general values of the parameters.[1]

Problem (1) is a purely mathematical one. It is closely related with the existence of algebraic relations between a few hypergeometric functions with values of parameters differing by an integer, the so-called "contiguous relations" [8]. The systematic procedure for solving the relevant recursion relation is based on the Gröbner basis technique. In particular, a proper solution for generalized hypergeometric functions, the so-called "differential reduction algorithm," was developed by Takayama [9]. (See Ref. [10] for a review.) By a differential reduction algorithm, we will understand a relation of the type $F(\alpha \pm j, \vec{b}; z) = \Pi_{k=1}^{m} D(\alpha + k; \vec{b}) F(\alpha, \vec{b}; z)$, where $j, k$ are integers, $\vec{b}$ is a list of additional

[1] All these procedures coincide with standard techniques used in the analytical calculation of Feynman diagrams. [4, 5] It has long been expected that all Feynman diagrams can be represented by some class of hypergeometric functions. Now we can propose specifically that any Feynman diagrams can be associated with the Gelfand-Karpanov-Zelevinskii (GKZ or A- hypergeometric function) hypergeometric functions [6]. Let us recall that Lauricella's, Horns' and generalized hypergeometric functions occur as special cases of the GKZ-systems. For an introduction, we recommended Ref. [7].

parameters and $D$ is a differential operator of the form $D = A(\alpha; \vec{b}; z)\frac{d}{dz} + B(\alpha; \vec{b}; z)$.[2] For Gauss hypergeometric functions, the reduction algorithm was presented in Refs. [11, 12]. For generalized hypergeometric function, is it equivalent to the statement that any function ${}_pF_{p-1}(\vec{a}; \vec{b}; z)$ can be expressed as a linear combination of functions with arguments that differ from the original ones by an integer, ${}_pF_{p-1}(\vec{a}+\vec{m}; \vec{b}+\vec{k}; z)$, and the function's first $p-1$ derivatives:

$$
\begin{aligned}
& R_{p+1}(\vec{a},\vec{b},z) \quad {}_pF_{p-1}(\vec{a}+\vec{m};\vec{b}+\vec{k};z) = \\
& \quad \left\{ R_1(\vec{a},\vec{b},z) \left(\frac{d}{dz}\right)^{p-1} + \cdots + R_{p-1}(\vec{a},\vec{b},z)\,\frac{d}{dz} + R_p(\vec{a},\vec{b},z) \right\} {}_pF_{p-1}(\vec{a};\vec{b};z) \;, \qquad (1.1)
\end{aligned}
$$

where $\vec{m}, \vec{k}$ are lists of integers, the $R_i$ are polynomials in parameters $a_i, b_j$ and $z$.

Problem (2) arises in physics in the context of the analytical calculation of Feynman diagrams. The complete solution of this problem is still open. We will mention here some results in this direction derived by physicists. Let us recall that there are three different ways to describe special functions:

(i) as an integral of the Euler or Mellin-Barnes type,

(ii) by a series whose coefficients satisfy certain recurrence relations,

(iii) as a solution of a system of differential and difference equations (holonomic approach).

For functions of a single variable, all of these representations are equivalent, but some properties of the function may be more evident in one representation than another. These three different representations have led physicists to three different approaches to developing the $\varepsilon$-expansion of hypergeometric functions.

The Euler integral representation (i) was developed intensively by Davydychev, Tarasov and their collaborators [13] and the most impressive result was the construction of the all-order $\varepsilon$-expansion of Gauss hypergeometric functions in terms of Nielsen polylogarithms [14]. This type of Gauss hypergeometric function is related to one-loop propagator-type diagrams with arbitrary masses and momenta, two-loop bubble diagrams with arbitrary masses, and one-loop massless vertex-type diagrams.

The series representation (ii) is a very popular and intensively studied approach. The first results of this type were derived by David Broadhurst [15] for the so-called "single scale" diagrams, which are associated with on-shell calculations in QED or QCD, with further developments appearing subsequently in Ref. [16].[3] Particularly impressive results involving series representations were derived recently by Moch, Uwer, and Weinzierl in the

[2] An algorithm of differential reduction of generalized hypergeometric functions to a minimal set allows the calculation of any Feynman diagram that is expressible in terms of hypergeometric functions without any reference to integration by parts or the differential equation technique. The application of this algorithm to the calculation of Feynman diagrams will be presented in another publication.

[3] Some relations between the Mellin-Barnes representations and series representations of Feynman diagrams follow from the Smirnov-Tausk approach [21].

framework of the nested sum approach.[17, 18] Algorithms based on this approach have been implemented in computer code.[19] The series approach leads to algebraic relations between the analytic coefficients of the $\varepsilon$-expansion, but does not provide a way to obtain a reduction of the original hypergeometric function before expansion. Another limitation of this approach is that the parameters are restricted to integer values or special combinations of half-integer values (so-called "zero-balance" parameter sets).[4]

For approach (iii), obtaining iterated solutions to the proper differential equations associated with hypergeometric functions, the first results were obtained for Gauss hypergeometric functions expanded about integer values of parameters.[22] In Ref. [23], that result was extended to combinations of integer and half-integer values of parameters. An advantage of the iterated solution approach over the series approach is that it provides a more efficient way to calculate each order of the $\varepsilon$-expansion, since it relates each new term to the previously derived terms rather than having to work with an increasingly large collection of independent sums at each subsequent order.

The aim of the present paper is to apply approach (iii) to proving the following theorem:[5]

**Theorem 1**
*The all-order $\varepsilon$-expansion of a generalized hypergeometric function ${}_pF_{p-1}(\vec{A}+\vec{a}\varepsilon;\vec{B}+\vec{b}\varepsilon;z)$, where $\vec{A}$ and $\vec{B}$ are lists of integers, are expressible in terms of generalized polylogarithms (see Eq. 1.2) with coefficients that are ratios of polynomials.*

To be specific, this means that

$$P(\{a\},\{b\},z)\ {}_pF_{p-1}(\vec{A}+\vec{a}\varepsilon;\vec{B}+\vec{b}\varepsilon;z) = \sum R_{\vec{s}}(z)\ \mathrm{Li}_{\vec{s}}(z)\,\varepsilon^k\ ,$$

where $\vec{s}=(s_1,\cdots,s_l)$ is a multiple index and $P(\{a\},\{b\},z), R_{\vec{s}}(z)$ are polynomials. The *generalized polylogarithms* are defined by the equation

$$\mathrm{Li}_{k_1,k_2,\cdots,k_n}(z) = \sum_{m_1>m_2>\cdots m_n>0} \frac{z^{m_1}}{m_1^{k_1} m_2^{k_2}\cdots m_n^{k_n}}\ , \tag{1.2}$$

For completeness, we recall that *generalized polylogarithms* (1.2) can be expressed as iterated integrals of the form

$$\mathrm{Li}_{k_1,\cdots,k_n}(z) = \int_0^z \underbrace{\frac{dt}{t}\circ\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{k_1-1\ \text{times}}\circ\frac{dt}{1-t}\circ\cdots\circ\underbrace{\frac{dt}{t}\circ\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{k_n-1\ \text{times}}\circ\frac{dt}{1-t}\ , \tag{1.3}$$

where, by definition

$$\int_0^z \underbrace{\frac{dt}{t}\circ\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{k_1-1\ \text{times}}\circ\frac{dt}{1-t} = \int_0^z \frac{dt_1}{t_1}\int_0^{t_1}\frac{dt_2}{t_2}\cdots\int_0^{t_{k-2}}\frac{dt_{k_1-1}}{t_{k_1-1}}\int_0^{t_{k_1-1}}\frac{dt_{k_1}}{1-t_{k_1}}\ . \tag{1.4}$$

[4]For some new results on $\varepsilon$-expansions of hypergeometric functions with nonzero-balance parameter sets of parameters (specifically, one half-integer parameter), see Ref. [20].

[5]In fact, the result expressed in this theorem can be proved within the nested sum approach [17]. However, our idea is to extend the iterated solution approach to this more complicated system and in the process, derive a more efficient algorithm for calculating the analytical coefficients of the $\varepsilon-$expansion.

The integral (1.3) is an iterated Chen integral [27] w.r.t. the differential forms $\omega_0 = dz/z$ and $\omega_1 = \frac{dz}{1-z}$, so that

$$\mathrm{Li}_{k_1,\cdots,k_n}(z) = \int_0^z \omega_0^{k_1-1}\omega_1 \cdots \omega_0^{k_n-1}\omega_1 \; . \tag{1.5}$$

## 2. All-order $\varepsilon$-expansion of generalized hypergeometric functions with integer values of parameters

In this section, we shall prove **Theorem 1**. We begin by noting that Eq. (1.1) can be written in a slightly different form: in terms of any basic function ${}_pF_{p-1}(\vec{a};\vec{b};z)$ and its first $p-1$ derivatives,

$$\begin{aligned} & R_{p+1}(\vec{a},\vec{b},z)\; {}_pF_{p-1}(\vec{a}+\vec{m};\vec{b}+\vec{k};z) = \\ & \left\{ R_1(\vec{a},\vec{b},z)\,\theta^{p-1} + \cdots + R_{p-1}(\vec{a},\vec{b},z)\,\theta + R_p(\vec{a},\vec{b},z) \right\} {}_pF_{p-1}(\vec{a};\vec{b};z) \; , \end{aligned} \tag{2.1}$$

where $\vec{m},\vec{k}$ are lists of integers, the $R_i$ are polynomials in parameters $a_i, b_j$ and $z$, and $\theta = z\frac{d}{dz}$ . The essential step in proving **Theorem 1** is the following lemma:

**Lemma 1**
*The all-order $\varepsilon$-expansion of the function ${}_pF_{p-1}(\vec{a}\varepsilon;\vec{1}+\vec{b}\varepsilon;z)$, is expressible in terms of generalized polylogarithms (Eq. 1.2).*

Lemma 1 could be proved in the same manner as in case of multiple (inverse) binomial sums. This was done in Ref. [17]. However, it is fruitful to prove it using the construction of an iterated solution of the proper differential equation related to the hypergeometric function.[6] We will follow this technique here, and in the process construct an iterative algorithm determining the analytical coefficients of the epsilon expansion.

Let us consider the differential equation for the hypergeometric function $\omega(z) = {}_pF_{p-1}(\vec{a}\varepsilon;\vec{1}+\vec{b}\varepsilon;z)$:

$$\left[ z\prod_{i=1}^{p}(\theta + a_i\varepsilon) - \theta\prod_{i=1}^{p-1}(\theta + b_i\varepsilon) \right] \omega(z) = 0 \; . \tag{2.2}$$

The boundary conditions for basis functions are $\omega(0) = 1$ and $\theta^j\omega(z)\big|_{z=0} = 0$, where $j = 1,\cdots,p-1$. The proper differential equation for $\omega(z)$ is valid in each order of $\varepsilon$. Defining the coefficients functions $w_k(z)$ at each order by

$$\omega(z) = \sum_{k=0}^{\infty} w_k(z)\varepsilon^k, \tag{2.3}$$

[6] The proper solution for Gauss hypergeometric functions was constructed in Refs. [22, 23].

the boundary conditions for the coefficient functions are

$$
\begin{aligned}
w_0(z) &= 1\ , && && (2.4a)\\
w_k(z) &= 0\ , && k<0\ , && (2.4b)\\
w_k(0) &= 0\ , && k\geq 1\ , && (2.4c)\\
z\tfrac{d}{dz}w_k(z)\big|_{z=0} &= 0\ , && k\geq 0\ , && (2.4d)\\
\cdots & && &&\\
\left(z\tfrac{d}{dz}\right)^{p-1}w_k(z)\Big|_{z=0} &= 0\ , && k\geq 0\ . && (2.4e)
\end{aligned}
$$

The differential equation (2.2) has the form

$$
\left[(1-z)\frac{d}{dz}\right]\left(z\frac{d}{dz}\right)^{p-1}w_k(z)=\sum_{i=1}^{p-1}\left[P_i(\vec{a})-\frac{1}{z}Q_i(\vec{b})\right]\left(z\frac{d}{dz}\right)^{p-i}w_{k-i}(z)+P_p(\vec{a})w_{k-p}(z)\ ,
$$

where $P_j(\vec{a})$ and $Q_j(\vec{b})$ are polynomials of order $j$ defined on spaces of $p$- and $(p–1)$-vectors $\vec{a}$ and $\vec{b}$, respectively. They are defined as

$$
\begin{aligned}
P_0 &= Q_0 = 1\ , && (2.5a)\\
P_r &= \Sigma^p_{i_1,\cdots,i_r=1}\Pi_{i_1<\cdots<i_r}a_{i_1}\cdots a_{i_r}\ , \quad r=1,\cdots,p\ , && (2.5b)\\
Q_r &= \Sigma^{p-1}_{i_1,\cdots,i_r=1}\Pi_{i_1<\cdots<i_r}b_{i_1}\cdots b_{i_r}\ , \quad r=1,\cdots,p-1\ , && (2.5c)\\
Q_p &= 0\ , && (2.5d)
\end{aligned}
$$

so that

$$
\begin{aligned}
P_1 &= \Sigma^p_{j=1}a_j\ , & Q_1 &= \Sigma^{p-1}_{j=1}b_j\ ,\\
P_2 &= \Sigma^p_{i,j=1;i<j}a_ia_j\ , & Q_2 &= \Sigma^{p-1}_{i,j=1;i<j}b_ib_j\ ,\\
P_3 &= \Sigma^p_{i_1,i_2,i_3=1;i_1<i_2<i_3}a_{i_1}a_{i_2}a_{i_3}\ , & Q_3 &= \Sigma^{p-1}_{i_1,i_2,i_3=1;i_1<i_2<i_3}b_{i_1}b_{i_2}b_{i_3}\ .\\
&\cdots & &\cdots \qquad\qquad (2.6)\\
& & Q_{p-1} &= \Pi^{p-1}_{i=1}b_i\ ,\\
P_p &= \Pi^p_{i=1}a_i\ , & Q_p &= 0.
\end{aligned}
$$

The polynomials $P_j$ and $Q_j$ satisfy the following relations:

$$
P_j(\vec{a},b)\ =\ P_j(\vec{a})+bP_{j-1}(\vec{a})\ ,\quad Q_j(\vec{a},b)=Q_j(\vec{a})+bQ_{j-1}(\vec{a}),\quad j=1,\cdots,p\ . \quad (2.7)
$$

In particular,

$$
P_j(\vec{a},0)=P_j(\vec{a})\ ,\quad Q_j(\vec{a},0)=Q_j(\vec{a})\ .
$$

Let us introduce a set of a new functions $\rho^{(j)}(z), j=1,\cdots,p-1$ defined by

$$
\rho^{(j)}(z)\ =\ \theta^j\omega(z)\equiv\left(z\frac{d}{dz}\right)^j\omega(z)=\sum_{k=0}^{\infty}\rho^{(j)}_k(z)\varepsilon^k\ ,\quad j=1,\cdots,p-1\ , \qquad (2.8)
$$

where the coefficient functions $\rho_k^{(j)}(z)$ satisfy

$$\rho_k^{(j)}(z) = \left(z\frac{d}{dz}\right)^j w_k(z) \; , \quad j = 1, \cdots, p-1 \; . \tag{2.9}$$

The boundary conditions for these new functions follow from Eq. (2.4e):

$$\rho_k^{(j)}(0) = 0 \; , \qquad k \geq 0 \; , \quad j \geq 1 \; . \tag{2.10}$$

Eq. (2.5) can be rewritten as a system of first-order differential equations

$$z\frac{d}{dz}\rho_k^{(j)}(z) \; = \; \rho_k^{(j+1)}(z) \; , \quad j = 0, 1, \cdots, p-1 \tag{2.11a}$$

$$(1-z)\frac{d}{dz}\rho_k^{(p-1)}(z) \; = \; \sum_{i=1}^{p}\left[P_i(\vec{a}) - \frac{1}{z}Q_i(\vec{b})\right]\rho_{k-i}^{(p-i)}(z) \; , \tag{2.11b}$$

and we have

$$w_k(z) \equiv \rho_k^{(0)}(z) \; . \tag{2.12}$$

The solution of system (2.11) can be presented in an iterated form:

$$\begin{aligned} \rho_k^{(p-1)}(z) \; &= \; \sum_{i=1}^{p}\left[P_i(\vec{a}) - Q_i(\vec{b})\right]\int_0^z \frac{dt}{1-t}\rho_{k-i}^{(p-i)}(t) \\ &- \sum_{i=1}^{p-2} Q_i(\vec{b})\rho_{k-i}^{(p-i-1)}(z) - Q_{p-1}(\vec{b})[w_{k-p+1}(z) - \delta_{0,k-p+1}] \; , \end{aligned} \tag{2.13a}$$

$$\rho_k^{(j-1)}(z) \; = \; \int_0^z \frac{dt}{t}\rho_k^{(j)}(t) \; , \quad k \geq 1 \; , \quad j = 1, 2, \cdots, p-1 \; , \tag{2.13b}$$

where $\delta_{a,b}$ is the Kronecker delta function.

From the system of Eq. (2.13), it is easy to find that

$$\rho_k^{(j)}(z) = 0, \quad k < p; \quad j = 0, 1, \cdots, p-1. \tag{2.14}$$

The first nonzero terms are generated by Eq. (2.13a) for $i = k = p$. Substituting this result into Eq. (2.13b) we will find the solution of the first iteration:

$$\rho_p^{(p-1-j)}(z) = P_p(\vec{a})\mathrm{Li}_{1+j}(z) \; , \quad j = 0, 1, \cdots, p-1, \tag{2.15}$$

where $\mathrm{Li}_j(z)$ is a classical polylogarithm [24] and $\mathrm{Li}_1(z) = -\ln(1-z)$. **Lemma 1** follows from the representation (2.11b), the value $w_0(z) = 1$, the definition of generalized polylogarithms (1.2), and Eq. (2.15).

The case when one of the upper parameters of the hypergeometric function is a positive integer number, ${}_pF_{p-1}(I_1, \vec{A} + \vec{a}\varepsilon; \vec{B} + \vec{b}\varepsilon; z)$, corresponds to $a_1$ equal to zero. A smooth limit exists in this case and the particular result can be reproduced from expression (2.13). **Theorem 1** is thus proved.

## 3. Explicit expressions for the first five coefficients of the expansion

Let us return to Eq. (2.13) and look at the next terms of the expansion. The first nonzero term of the iteration is given by Eq. (2.14). The second iteration corresponds to $k = p+1$. In the r.h.s. of Eq. (2.13a), only terms with $i = 1$ produce a non-zero contribution,

$$\frac{\rho_{p+1}^{(p-1)}(z)}{P_p} = \Delta_1 \frac{1}{2} \ln^2(1-z) - Q_1 \mathrm{Li}_2(z) \ ,$$

where for simplicity, we omit arguments in the functions $P_j, Q_j$ and introduce a notation

$$\Delta_j = P_j - Q_j \ , \quad j = 1, \cdots, p-1.$$

Substituting the results in Eq. (2.13b), we will get the solution of the second iteration:

$$\frac{\rho_{p+1}^{(p-1-j)}(z)}{P_p} = \Delta_1 \mathrm{Li}_{j+1,1}(z) - Q_1 \mathrm{Li}_{2+j}(z) \ , \quad j = 0, 1, \cdots, p-1, \tag{3.1}$$

where $\mathrm{Li}_{a_1,\cdots,a_k}(z)$ is a generalized polylogarithm. The third iteration corresponds to $k = p+2$, and in the r.h.s. of Eq. (2.13a) only terms with $i = 1, 2$ will produce a non-zero contribution,

$$\begin{aligned} \frac{\rho_{p+2}^{(p-1-j)}(z)}{P_p} &= \Delta_1^2 \mathrm{Li}_{j+1,1,1}(z) + (\Delta_2 - Q_1\Delta_1)\, \mathrm{Li}_{j+1,2}(z) \\ &+ \left(Q_1^2 - Q_2\right) \mathrm{Li}_{j+3}(z) - Q_1 \Delta_1 \mathrm{Li}_{j+2,1}(z) \ , \quad j = 0, 1, \cdots, p-1. \end{aligned} \tag{3.2}$$

The fourth iteration corresponds to $k = p+3$ and equal to

$$\begin{aligned} \frac{\rho_{p+3}^{(p-1-j)}(z)}{P_p} &= \Delta_1^3 \mathrm{Li}_{j+1,1,1,1}(z) + \Delta_1 (\Delta_2 - Q_1\Delta_1) \left[\mathrm{Li}_{j+1,1,2}(z) + \mathrm{Li}_{j+1,2,1}(z)\right] \\ &+ \left(\Delta_1 Q_1^2 - \Delta_1 Q_2 - \Delta_2 Q_1 + \Delta_3\right) \mathrm{Li}_{j+1,3}(z) - Q_1 \Delta_1^2 \mathrm{Li}_{j+2,1,1}(z) \\ &+ Q_1 (\Delta_1 Q_1 - \Delta_2)\, \mathrm{Li}_{j+2,2}(z) + \Delta_1 \left(Q_1^2 - Q_2\right) \mathrm{Li}_{j+3,1}(z) \\ &- \left(Q_1^3 - 2Q_1Q_2 + Q_3\right) \mathrm{Li}_{j+4}(z) \ , \quad j = 0, 1, \cdots, p-1. \end{aligned} \tag{3.3}$$

The fifth iteration corresponds to $k = p+4$ and equal to

$$\begin{aligned} \frac{\rho_{p+4}^{(p-1-j)}(z)}{P_p} &= \Delta_1^4 \mathrm{Li}_{j+1,1,1,1,1}(z) + \left(\Delta_1^2 Q_1^2 - 2\Delta_1\Delta_2 Q_1 + \Delta_2^2\right) \mathrm{Li}_{j+1,2,2}(z) \\ &+ \Delta_1^2 (\Delta_2 - Q_1\Delta_1) \left[\mathrm{Li}_{j+1,1,1,2}(z) + \mathrm{Li}_{j+1,1,2,1}(z) + \mathrm{Li}_{j+1,2,1,1}(z)\right] \\ &+ \Delta_1 \left\{\Delta_1 \left(Q_1^2 - Q_2\right) - \Delta_2 Q_1 + \Delta_3\right\} \left[\mathrm{Li}_{j+1,1,3}(z) + \mathrm{Li}_{j+1,3,1}(z)\right] \\ &- Q_1 \Delta_1^3 \mathrm{Li}_{j+2,1,1,1}(z) + Q_1 \Delta_1 (\Delta_1 Q_1 - \Delta_2) \left[\mathrm{Li}_{j+2,1,2}(z) + \mathrm{Li}_{j+2,2,1}(z)\right] \\ &+ \Delta_1^2 \left(Q_1^2 - Q_2\right) \mathrm{Li}_{j+3,1,1}(z) - \Delta_1 \left(Q_1^3 - 2Q_1Q_2 + Q_3\right) \mathrm{Li}_{j+4,1}(z) \\ &+ Q_1 \left\{\Delta_1 \left(Q_2 - Q_1^2\right) + \Delta_2 Q_1 - \Delta_3\right\} \mathrm{Li}_{j+2,3}(z) \\ &+ \left[Q_1 \left\{\Delta_1 \left(Q_2 - Q_1^2\right) + \Delta_2 Q_1\right\} - Q_2 \Delta_2\right] \mathrm{Li}_{j+3,2}(z) \\ &+ \left(Q_1^4 - 3Q_1^2 Q_2 + 2Q_1Q_3 + Q_2^2 - Q_4\right) \mathrm{Li}_{j+5}(z) \\ &+ \left\{\Delta_4 - Q_1\Delta_3 + \Delta_2 \left(Q_1^2 - Q_2\right) - \Delta_1 \left(Q_1^3 - 2Q_1Q_2 + Q_3\right)\right\} \mathrm{Li}_{j+1,4}(z) \ , \\ & j = 0, 1, \cdots, p-1. \end{aligned} \tag{3.4}$$

For lower values of the index $p$, the following relations can be used for transforming harmonic polylogarithms [26] to the classical [24] or Nielsen [25] ones:

$$\mathrm{Li}_{j,\underbrace{1,1,\cdots,1}_{p\ \mathrm{times}}}(z) = \mathrm{S}_{j-1,p+1}(z) \ , \tag{3.5a}$$

$$\mathrm{S}_{0,j}(z) = \frac{(-1)^j}{j!}\ln^j(1-z) \ , \tag{3.5b}$$

$$\mathrm{Li}_{1,2}(z) = -\ln(1-z)\mathrm{Li}_2(z) - 2\mathrm{S}_{1,2}(z) \ , \tag{3.5c}$$

$$\mathrm{Li}_{1,3}(z) = -\ln(1-z)\mathrm{Li}_3(z) - \frac{1}{2}\left[\mathrm{Li}_2(z)\right]^2 \ , \tag{3.5d}$$

$$\mathrm{Li}_{1,4}(z) = -\ln(1-z)\mathrm{Li}_4(z) + F_2(z) \ , \tag{3.5e}$$

$$\mathrm{Li}_{2,2}(z) = \frac{1}{2}\left[\mathrm{Li}_2(z)\right]^2 - 2\mathrm{S}_{2,2}(z) \ , \tag{3.5f}$$

$$\mathrm{Li}_{3,2}(z) = \frac{1}{2}\mathrm{Li}_2(z)\,\mathrm{Li}_3(z) + \frac{1}{2}F_2(z) - 2\mathrm{S}_{3,2}(z) \ , \tag{3.5g}$$

$$\mathrm{Li}_{2,3}(z) = -\frac{3}{2}F_2(z) - \frac{1}{2}\mathrm{Li}_2(z)\,\mathrm{Li}_3(z) \ . \tag{3.5h}$$

$$\mathrm{Li}_{1,1,2}(z) = \frac{1}{2}\ln^2(1-z)\mathrm{Li}_2(z) + 2\ln(1-z)\mathrm{S}_{1,2}(z) + 3\mathrm{S}_{1,3}(z) \ , \tag{3.5i}$$

$$\mathrm{Li}_{1,2,1}(z) = -\ln(1-z)\mathrm{S}_{1,2}(z) - 3\mathrm{S}_{1,3}(z) \ , \tag{3.5j}$$

$$\mathrm{Li}_{2,2,1}(z) + \mathrm{Li}_{2,1,2}(z) = F_1(z) - \mathrm{Li}_2(z)\,\mathrm{S}_{1,2}(z) \ , \tag{3.5k}$$

$$\begin{aligned}\mathrm{Li}_{1,1,3}(z) + \mathrm{Li}_{1,3,1}(z) &= \frac{1}{2}\ln^2(1-z)\mathrm{Li}_3(z) + \frac{1}{2}\ln(1-z)\left[\mathrm{Li}_2(z)\right]^2 \\ &\quad - \ln(1-z)\mathrm{S}_{2,2}(z) - \mathrm{Li}_2(z)\,\mathrm{S}_{1,2}(z) + F_1(z) \ ,\end{aligned} \tag{3.5l}$$

$$\begin{aligned}\mathrm{Li}_{1,2,2}(z) &= \ln(1-z)\left\{2\mathrm{S}_{2,2}(z) - \frac{1}{2}\left[\mathrm{Li}_2(z)\right]^2\right\} \\ &\quad + 2\mathrm{Li}_2(z)\,\mathrm{S}_{1,2}(z) - 2F_1(z) \ ,\end{aligned} \tag{3.5m}$$

$$\begin{aligned}\mathrm{Li}_{1,1,1,2}(z) + \mathrm{Li}_{1,1,2,1}(z) + \mathrm{Li}_{1,2,1,1}(z) &= -\frac{1}{6}\ln^3(1-z)\mathrm{Li}_2(z) \\ &\quad -\frac{1}{2}\ln^2(1-z)\mathrm{S}_{1,2}(z) - \ln(1-z)\mathrm{S}_{1,3}(z) - 2\mathrm{S}_{1,4}(z) \ ,\end{aligned} \tag{3.5n}$$

where we have introduced two new functions related algebraically (see Eqs. (2.23) – (2.25) in Ref. [23]):

$$F_1(z) = \int_0^z \frac{dx}{x}\ln^2(1-x)\mathrm{Li}_2(x) \ , \tag{3.6}$$

$$F_2(z) = \int_0^z \frac{dx}{x}\ln(1-x)\mathrm{Li}_3(x) \ . \tag{3.7}$$

For completeness, we will present the values of $P$ and $Q$ for $p = 3, 4$:

- $p = 3$

$$\begin{aligned} P_1(\vec{a}) &= a_1 + a_2 + a_3 \ , \qquad & Q_1(\vec{b}) &= b_1 + b_2 \ , \\ P_2(\vec{a}) &= a_1a_2 + a_1a_3 + a_2a_3 \ , \qquad & Q_2(\vec{b}) &= b_1b_2 \ . \\ P_3(\vec{a}) &= a_1a_2a_3 \ , \qquad & Q_3(\vec{b}) &= 0 \ . \end{aligned} \tag{3.8}$$

- $p=4$

$$
\begin{aligned}
P_1(\vec{a}) &= a_1+a_2+a_3+a_4\ , & Q_1(\vec{b}) &= b_1+b_2+b_3\ , \\
P_2(\vec{a}) &= a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4\ , & Q_2(\vec{b}) &= b_1b_2+b_1b_3+b_2b_3\ . \\
P_3(\vec{a}) &= a_1a_2a_3+a_1a_2a_4+a_2a_3a_4\ , & Q_3(\vec{b}) &= b_1b_2b_3\ . \\
P_4(\vec{a}) &= a_1a_2a_3a_4\ , & Q_4(\vec{b}) &= 0\ .
\end{aligned} \tag{3.9}
$$

The first few coefficients, up to order 4, could be cross-checked using the results of Ref. [28].

We would like to point out that Eqs. (3.1) – (3.4) contain an explicit logarithmic singularity at $z=1$. It is well-known that the generalized hypergeometric function ${}_pF_{p-1}(\vec{a};\vec{b};z)$ converges absolutely on the unit circle $|z|=1$ if

$$
\mathrm{Re}\left(\sum_{j=1}^{p-1} b_j - \sum_{j=1}^{p} a_j\right) > 0.
$$

In this case, the coefficients of the $\varepsilon$-expansion also converge at each order in $\varepsilon$. To get a smooth limit, it is enough to rewrite Eqs. (3.1) – (3.4) in terms of functions of argument $1-z$ and set $z=1$.

## 4. Conclusions

We have shown (**Theorem 1**) that the $\varepsilon$-expansions of generalized hypergeometric functions with integer values of parameters are expressible in terms of generalized polylogarithms (see Eq. (1.2)) with coefficients that are ratios of polynomials. The proof includes (i) the differential reduction algorithm; and (ii) iterative algorithms for calculating the analytical coefficients of the $\varepsilon$-expansion of basic hypergeometric functions (see Eq. (2.13)). The first five coefficients of the $\varepsilon$-expansion for basis hypergeometric functions are calculated explicitly in Eqs. (2.15), (3.1), (3.2), (3.3), and (3.4). The FORM [29] representations of these expressions and the next coefficients are available via Ref. [30].

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